FLEXURAL ANALYSIS OF RECTANGULAR THIN PLATE USING IMPROVED FINITE DIFFERENCE METHOD

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Abstract: **This research developed the mathematical model for deflection in rectangular thin plate. The improved finite difference model for deflection was developed using Taylor's series expression. The model was applied in analyzing the flexural behaviour of rectangular thin plate with all edges clamped. The results obtained from this numerical method were compared with the ordinary finite difference method and the exact solution to check the accuracy of the solution. Two cases of plate were considered. The results emerging from plate with 5*5 interior nodal point showed an average percentage error of -4.82% while that of plate with 7*7 nodal point showed +13.92% from the exact solution. Hence the results suggest that the accuracy of the model lies within the plate with 5*5 nodal point. However the convergence to the exact solution was close and rapid. Thus the use of improved finite difference method proved to be more accurate than the ordinary finite difference method. Therefore the model developed is effective and is recommended for use in structural engineering.**

Keywords: **Improved Finite Difference, Taylor Series, Rectangular thin plate, Mathematical model, Convergence, Exact solution.**

LIST OF SYMBOLS

'D' is the flexural rigidity, $'p_z'$ is the applied transverse load,$'v'$ is the passion ratio, $'E'$ is the modulus of elasticity and $'h'$ is the plate thickness. $f^{(k)}(x_0)$ = the K^{th} derivative of $f(x)$ at the point $x = x_0$, $E_n(x - x_0)$ = truncation error which accounts for the neglected term of the series, $\Delta x =$ distance along the x-axis, $\alpha =$ aspect ratio, $\Delta y =$ distance along y-axis, $m =$ length per each mesh and $n =$ width per each mesh.

1. INTRODUCTION

Analysis or solution to a plate problem means the determination of displacement, moment, stress etc at various points of the plate. Thus, plate analysis ensures the stability of the plate to resist the design load (Roknuzzaman et al, 2015). Plate bending refers to the deflection of a plate perpendicular to the plane of the plate under the action of external forces and moments. The amount of deflection can be determined by solving the differential equations of the plate theory. Thus the stresses in the plate can be calculated from the deflections (Baglekar and Deshmukh, 2014).

A number of research works were performed on the solution of plate problems based on analytical and numerical methods in the past years. The analytical methods search for the universal mathematical expressions representing the general and exact solution of a plate problem. Unfortunately, analytical solutions has restrictions in areas of practical interest therefore the numerical method were recommended to provide approximate solutions in the form of numbers, to the mathematical equations governing a plate. (Ezeh et al, 2013). Ventsel (2001) and Ezeh et al, (2013) used ordinary finite difference method to get the solution for pure bending analysis of thin rectangular flat plate having various boundary conditions. From the research works, it was indicated that the convergence characteristics of the ordinary finite difference method converge relatively fast towards the exact solution of a given plate problem, provided the finite difference mesh point is not too small. That is, when using very small subdivisions, the convergence becomes quite slow (Ezeh et al, 2013). Furthermore, when higher order derivatives and a large number of mesh points are involved, the solution due to machine errors may converge to a wrong number because the polynomials used in deriving the ordinary finite difference expression agree only in value with the exact function at the mesh points. Thus a small division of mesh will result in a large number of simultaneous equations thereby creating round-off errors in the computer solution which adversely affect its accuracy. Therefore when higher accuracy in the finite difference solution of plate problems is required, the improved finite difference technique is recommended (Szilard, 2004). Szilard (2004) applied the improved finite difference method to determine the deflection on pure bending analysis of thin square plate with uniformly distributed load and boundary conditions, CCSS. Furthermore Ibearugbulem et al (2014) applied the improved finite difference method to obtain the pure bending analysis of one dimensional line continuum with various boundaries conditions.

In the present work, a rectangular thin plate clamped at all four edges (CCCC) is analysed using the improved finite difference method and the results is compared with the ordinary finite difference solutions and the exact solutions.

2. LITERATURE REVIEW

2.1 THIN PLATE THEORY:

The governing differential equation for deflection of thin plates was given by Szilard (2004) as

$$D\left[\frac{d^4w}{dx^4} + 2\frac{d^4w}{dx^2\,dy^2} + \frac{d^4\,w}{dy^4}\right] = P_z(x,y) \qquad (2.1)$$

where

$$D = \frac{Eh^3}{12(1-v^2)} \qquad (2.2)$$

(Kirchhoff, 1876; Timoshenko, 1956; Ugural, 1999; Ventsel, 2001; Szilard, 2004; Reddy, 2007).

2.2 TAYLOR SERIES:

Most numerical methods are based on Taylor series expansion which is given as

$$T(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \qquad (2.1)$$

Thus in summation notation form will be

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!} + E_n\,(x-x_0) \qquad (2.2)$$

(Szilard, 2004; Math-image, 2013; Ibearugbulem et al, 2014).

3. MATERIALS AND METHOD

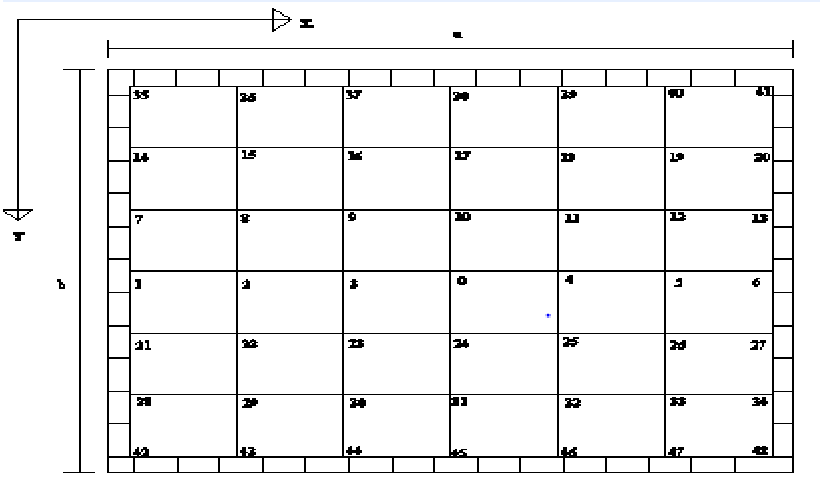

Figure 3.1: Diagram showing equally spaced Discretized Rectangular Plate

3.1 FORMULATING THE EXPRESSION FOR THE IMPROVE FINITE DIFFERENCE ALONG THE X AND Y-AXES

Considering the discretized plate as shown in fig (3.1) and taking f_o as the pivoted point. The displacement of each node will be obtained using central difference method. Thus expanding the function f_3 and f_4 along x-axes and f_{10} and f_{24} along y-axes using Taylor series and truncating at the fourth-order with step size $(\Delta x) = h$ will give the following equations;

For x-axes

$$f_4 = f_o + \frac{h}{1} f_o' + \frac{(h)^2}{2} f_o'' + \frac{(h)^3}{6} f_o''' + \frac{(h)^4}{24} f_o^{iv} \quad (3.1)$$

$$f_3 = f_o - \frac{h}{1} f_o' + \frac{(h)^2}{2} f_o'' - \frac{(h)^3}{6} f_o''' + \frac{(h)^4}{24} f_o^{iv} \quad (3.2)$$

$$f_5 = f_o + 2hf_o' + 2h^2 f_o'' - \left(\frac{4h^3}{3}\right) f_o''' + \left(\frac{2h^4}{3}\right) f_o^{iv} \quad (3.3)$$

$$f_2 = f_o - 2hf_o' + 2h^2 f_o'' - \frac{4h^3}{3} f_o''' + \frac{2h^4}{3} f_o^{iv} \quad (3.4)$$

For y-axes

$$f_{10} = f_o + \frac{h}{1} f_o' + \frac{(h)^2}{2} f_o'' + \frac{(h)^3}{6} f_o''' + \frac{(h)^4}{24} f_o^{iv} \quad (3.5)$$

$$f_{24} = f_o - \frac{h}{1} f_o' + \frac{(h)^2}{2} f_o'' - \frac{(h)^3}{6} f_o''' + \frac{(h)^4}{24} f_o^{iv} \quad (3.6)$$

$$f_{17} = f_o + 2hf_o' + 2h^2 f_o'' - \left(\frac{4h^3}{3}\right) f_o''' + \left(\frac{2h^4}{3}\right) f_o^{iv} \quad (3.7)$$

$$f_{31} = f_o - 2hf_o' + 2h^2 f_o'' - \frac{4h^3}{3} f_o''' + \frac{2h^4}{3} f_o^{iv} \quad (3.8)$$

Subtracting equation (3.1) from (3.2), (3.5) from (3.6) and addition equation (3.1) and (3.2), (3.5) and (3.6). Hence repeating the same procedure for equation (3.3), (3.4) and (3.7), (3.8) and making f' and f'' subject of the formula from the equations will give the first and second order derivative for the improved finite difference method. Following the same procedure above using the equations below

for x-axes

$$f_4' = f_o' + \frac{h}{1} f_o'' + \frac{(h)^2}{2} f_o''' + \frac{(h)^3}{6} f_o^{iv} + \frac{(h)^4}{24} f_o^{v} \quad (3.9)$$

$$f_3' = f_o' - \frac{h}{1} f_o'' + \frac{(h)^2}{2} f_o''' - \frac{(h)^3}{6} f_o^{iv} + \frac{(h)^4}{24} f_o^{v} \quad (3.10)$$

$$f_5' = f_o' + 2hf_o'' + 2h^2 f_o''' + \frac{4h^3}{3} f_o^{iv} + \frac{2h^4}{3} f_o^{v} \quad (3.11)$$

$$f_2' = f_o' - 2hf_o'' + 2h^2 f_o''' - \frac{4h^3}{3} f_o^{iv} + \frac{2h^4}{3} f_o^{v} \quad (3.12)$$

For y-axes

$$f_{10}' = f_o' + \frac{h}{1} f_o'' + \frac{(h)^2}{2} f_o''' + \frac{(h)^3}{6} f_o^{iv} + \frac{(h)^4}{24} f_o^{v} \quad (3.13)$$

$$f_{24}' = f_o' - \frac{h}{1} f_o'' + \frac{(h)^2}{2} f_o''' - \frac{(h)^3}{6} f_o^{iv} + \frac{(h)^4}{24} f_o^{v} \quad (3.14)$$

$$f'_{17} = f'_o + 2hf''_o + 2h^2 f'''_o + \frac{4h^3}{3} f_o^{iv} + \frac{2h^4}{3} f_o^{v} \qquad (3.15)$$

$$f'_{31} = f'_o - 2hf''_o + 2h^2 f'''_o - \frac{4h^3}{3} f_o^{iv} + \frac{2h^4}{3} f_o^{v} \qquad (3.16)$$

To determine the third derivative of the improved finite difference method while the fourth derivative is obtained using the following equations

For x-axes

$$f''_5 = f''_o + 2hf'''_o + 2h^2 f_o^{iv} + \frac{4h^3}{3} f_o^{v} + \frac{2h^4}{3} f_o^{vi} \qquad (3.17)$$

$$f''_2 = f''_o - 2hf'''_o + 2h^2 f_o^{iv} - \frac{4h^3}{3} f_o^{v} + \frac{2h^4}{3} f_o^{vi} \qquad (3.18)$$

For y-axes

$$f''_{10} = f''_o + \frac{h}{1} f'''_o + \frac{(h)^2}{2} f_o^{iv} + \frac{(h)^3}{6} f_o^{v} + \frac{(h)^4}{24} f_o^{vi} \qquad (3.19)$$

$$f''_{24} = f''_o - \frac{h}{1} f'''_o + \frac{(h)^2}{2} f_o^{iv} - \frac{(h)^3}{6} f_o^{v} + \frac{(h)^4}{24} f_o^{vi} \qquad (3.20)$$

Thus, the summary for the order of the improve finite difference expressions are shown in Table 3.1

Table 3.1 Summary of the High Order Improved Finite Difference Expression

Order of Derivatives	Order of Improved Finite Difference Expression
$\frac{dw}{dx}$	$\frac{1}{12h}(f_2 - 8f_3 + 8f_4 - f_5)$
$\frac{d^2w}{dx^2}$	$\frac{1}{12h^2}(-f_2 + 16f_3 - 30f_o + 16f_4 - f_5)$
$\frac{d^3w}{dx^3}$	$\frac{1}{24h^3}(f_1 - 16f_2 + 29f_3 + 0f_o - 29f_4 + 16f_5 - f_6)$
$\frac{d^4w}{dx^4}$	$\frac{1}{12h^4}(-f_1 + 18f_2 - 63f_3 + 92f_o - 63f_4 + 18f_5 - f_6)$
$\frac{dw}{dy}$	$\frac{1}{12h}(f_{31} - 8f_{24} + 8f_{10} - f_{17})$
$\frac{d^2w}{dy^2}$	$\frac{1}{12h^2}(-f_{31} + 16f_{24} - 30f_o + 16f_{10} - f_{17})$
$\frac{d^3w}{dy^3}$	$\frac{1}{24h^3}(f_{45} - 16f_{31} + 29f_{24} + 0f_o - 29f_{10} + 16f_{17} - f_{38})$
$\frac{d^4w}{dy^4}$	$\frac{1}{12h^4}(-f_{45} + 18f_{31} - 63f_{24} + 92f_o - 63f_{10} + 18f_{17} - f_{38})$

3.2 Formulating the Improved Finite Difference Expression for Deflection by the Method of Higher Approximation

The improved finite difference expression for deflection is formulated using equation (2.1), that is

$$\frac{d^4w}{dx^4} + 2\frac{d^4w}{dx^2dy^2} + \frac{d^4w}{dy^4} = \frac{P(x,y)}{D} \qquad (3.21)$$

Applying the derivative from table 3.1 the improved expression for deflection will give

$$\left[-\frac{f_1}{12h^4}+\frac{18f_2}{12h^4}-\frac{63f_3}{12h^4}+\frac{92f_0}{12h^4}-\frac{63f_4}{12h^4}+\frac{18f_5}{12h^4}-\frac{f_6}{12h^4}\right]$$
$$+\left[\frac{900f_0}{72h^4}-\frac{480f_{10}}{72h^4}-\frac{48f_{24}}{72h^4}+\frac{30f_{17}}{72h^4}+\frac{30f_{31}}{72h^4}-\frac{480f_4}{72h^4}+\frac{256f_{11}}{72h^4}+\frac{256f_{25}}{72h^4}-\frac{16f_{18}}{72h^4}-\frac{16f_{32}}{72h^4}-\frac{480f_3}{72h^4}\right.$$
$$+\frac{256f_9}{72h^4}+\frac{256f_{23}}{72h^4}-\frac{16f_{16}}{72h^4}-\frac{16f_{30}}{72h^4}+\frac{30f_5}{72h^4}-\frac{16f_{12}}{72h^4}-\frac{16f_{26}}{72h^4}+\frac{f_{19}}{72h^4}+\frac{f_{33}}{72h^4}+\frac{30f_2}{72h^4}-\frac{16f_8}{72h^4}-\frac{16f_{22}}{72h^4}$$
$$\left.+\frac{f_{15}}{72h^4}+\frac{f_{29}}{72h^4}\right]+\left[-\frac{f_{45}}{12h^4}+\frac{18f_{31}}{12h^4}-\frac{63f_{24}}{12h^4}+\frac{92f_0}{12h^4}-\frac{63f_{10}}{12h^4}+\frac{18f_{17}}{12h^4}-\frac{f_{38}}{12h^4}\right]$$
$$=\frac{P(x,y)}{D} \qquad (3.22)$$

According to Szilard (2004) when considering a rectangular mesh

$$\Delta x=\alpha(\Delta y)\ and\ \frac{m}{n}=\alpha=\frac{1}{\alpha} \qquad (3.23)$$

Thus considering equation (3.22) in the expression for deflection and simplifying further by collecting like terms and finding the L.C.M of each like term will give

$$\frac{1}{12h^4}[(92f_0+150\alpha^2f_0+92\alpha^4f_0)-(f_1)+(18f_2+5\alpha^2f_2)+(-63f_3-80\alpha^2f_3)+(-63f_4-80\alpha^2f_4)$$
$$+(18f_5+5\alpha^2f_5)+(-f_6)+(-80\alpha^2f_{10}-63\alpha^4f_{10})+(-80\alpha^2f_{24}-63\alpha^4f_{24})$$
$$+(5\alpha^2f_{17}-18\alpha^4f_{17})+(5\alpha^2f_{31}-18\alpha^4f_{31})+(42\alpha^2f_{11})+(42\alpha^2f_{25})-(2\alpha^2f_{18})-(2\alpha^2f_{32})$$
$$+(42\alpha^2f_9)+(42\alpha^2f_{23})-(2\alpha^2f_{16})-(2\alpha^2f_{30})-(2\alpha^2f_{12})-(2\alpha^2f_{26})-(2\alpha^2f_8)-(2\alpha^2f_{22})$$
$$-(\alpha^4f_{45})-(\alpha^4f_{38})]$$
$$=\frac{P(x,y)}{D} \qquad (3.24)$$

Therefore equation (3.24) shows the expression for the improved finite difference coefficient for deflection. The mathematical model of the improved finite difference for deflection is shown in figure 3.2

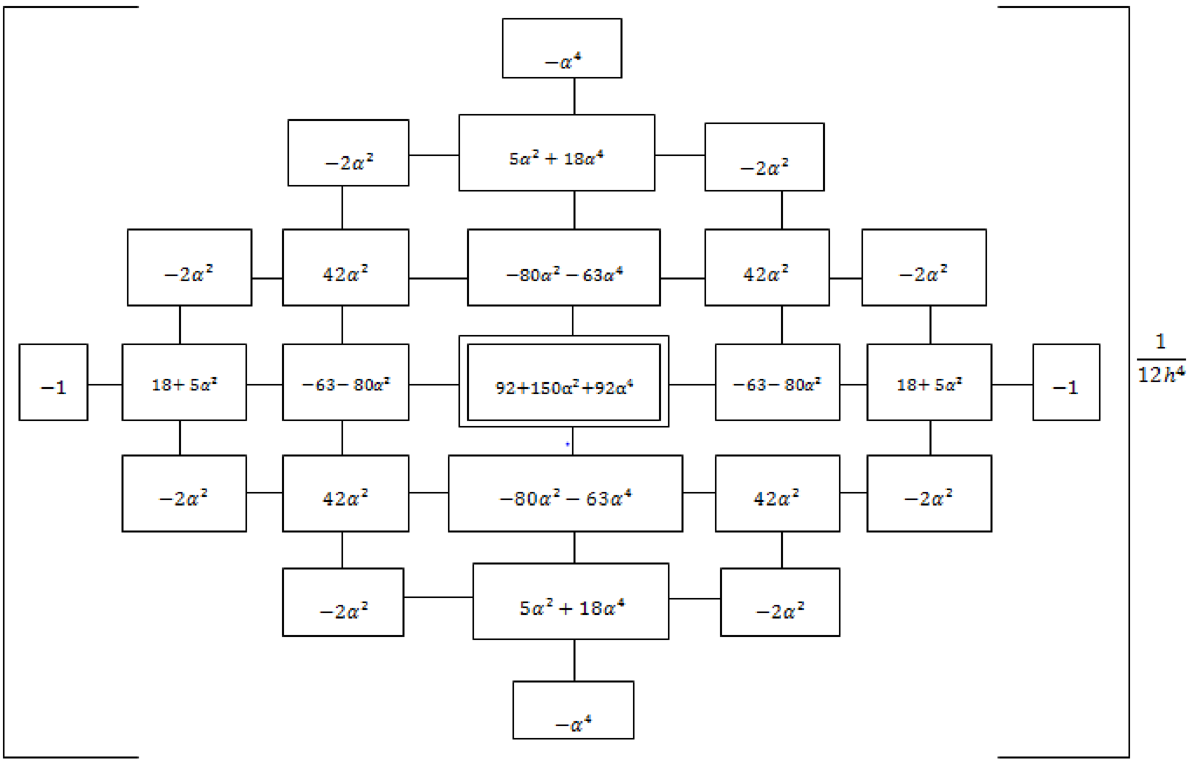

Figure 3.2 Mathematical Model of Deflection for Improve Finite Difference Metho

3.3 Boundary Conditions:

For a thin rectangular flat plate with all edge clamped having an edge length $'a'$ and $'b'$ there will be eight boundary conditions to be considered which are

$$(w)_x = 0, (\frac{dw}{dx})_{x=0,a} = 0, (w)_y = 0(\frac{dw}{dy})_{y=0,b} = 0 \qquad (3.25)$$

The solution of the governing plate equation by the improved finite difference method requires proper finite difference representation of the boundary conditions. Thus using central difference will introduce fictitious points outside of the plate boundaries. Therefore from the clamped boundary conditions the improved boundary conditions is derived.

4. NUMERICAL ANALYSIS

4.1 Generation of Nodes:

In generating the improved finite difference nodes two cases of plates were considered with a step size of

$$\Delta a = \frac{a}{6}\ and \Delta b = \frac{b}{6}, \Delta a = \frac{a}{8} \text{ and } \Delta b = \frac{b}{8} \qquad (4.1)$$

At each node there is one displacement component. There are total of 25 nodes (for plate with 5*5 nodal point) and 49 nodes (plate with 7*7 nodal point) on the plates. From the support consideration all the nodes along the clamped edges have zero deflection; these nodes are marked with zero value. Due to symmetry of loading all deflection component must have equal deflection and hence they are marked with same deflection notation. Doing this, in total 9 (plate with 5*5 nodal point) and 16 (plate with 7*7 nodal point) unknown deflection component are found for the plates.

4.2 Application and Result of the Improve Finite Difference:

Deflection is obtained by taking the inverse of the coefficient matrix and multiplying the inverse matrix with the right hand side of equation (4.2)

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} F1 \\ F2 \\ F3 \end{bmatrix} = \frac{P(\Delta L)^4}{D} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (4.2)$$

Where $A_{11} \dots . A_{33}$ is the coefficient matrix, $F1 \dots F3$ is the deflection coefficient, P is the uniform load, (ΔL) is the step size and D is the depth of the plate.

4.3 Graphical Representation of Results:

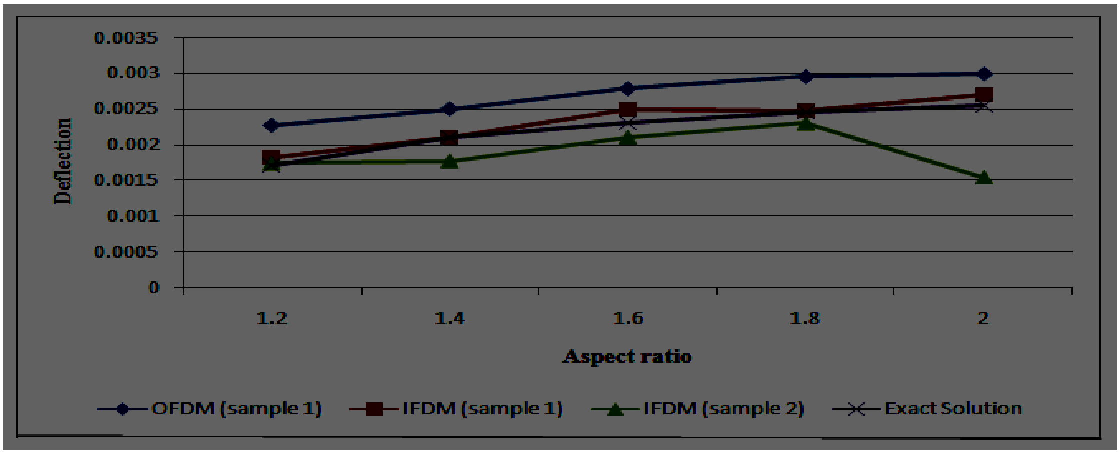

Figure 4.3: Comparison of the Numerical Solutions to the Exact Solution

Graph plotted for maximum deflections of various aspect ratios ranging from 1.2 to 2 for the numerical solution and exact solution as shown in figure 4.3, indicated that the results obtained for ordinary finite difference fell far above the exact solution. Hence plate with 5*5 interior nodal points of the improved finite difference results fell close to the exact solution while plate with 7*7 interior nodal points of the improved finite difference results fell below the exact solution.

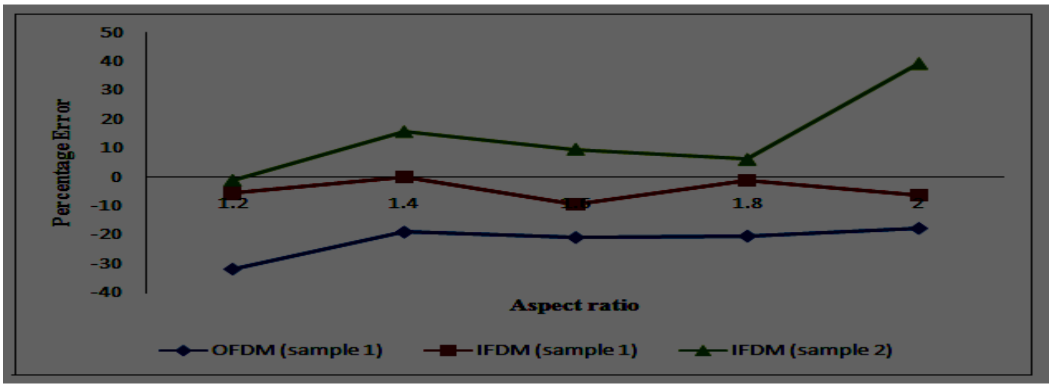

Figure 4.4: Percentage Error of the Numerical Solutions to the Exact Solution

Graph plotted for percentage error of the numerical solutions to the exact solution as shown in figure 4.4 , indicated that the improved finite difference results obtained from the plate with 5*5 interior nodal points approximated closely to the exact solution than the ordinary finite difference while the improved finite difference result obtained for plate with 7*7 interior nodal points overestimated the results to the exact solution.

5. CONCLUSIONS

The following conclusions were made

i. The analysis indicates that the maximum deflection of the plate occurred at the centre.

ii. It can be seen from the analysis that the results emerging from plate with sample 1 showed an average percentage error of -4.8% while that of plate with sample 2 shows +13.92% from the exact solution. Hence the result suggests that the accuracy of the model lies within sample 1 of the plate.

iii. The results show that the numerical factors for deflection increase as the span ratio increases which is in agreement with those of the exact approach (Timoshinko, 1987).

iv. The converged to the exact solution is close and rapid.

v. The use of improved finite difference by the method of higher approximation has proved to be more accurate than the ordinary finite difference method.

REFERENCES

[1] Ezeh, J.C, Ibearugbulem, O.M, Onyechere, C.I. (2013). Pure Bending Analysis of Thin Rectangular Flat Plates using Ordinary Finite Difference Method. International Journal of Emerging Technology and Advanced Engineering. Vol. 3, ISSN 2250-2459, ISO 9001:2008.

[2] Ibearugbulem O.M, Owus M, Osadebe, N. N, Ezeh, J. C, Onwuka, D.O (2011). Buckling Analysis of Axially Compressed SSSS Thin Rectangular Plate using Taylor-Mclaurin Shape Function. International Journal of Civil and Structural Engineering . Vol 2, No 2, ISSN 0976-4399.

[3] Md. Roknuzzaman, Md. Belal Hossain, Md. Rashedul Haque, Dr. Tarif Uddin Ahmed (2015). Analysis of Rectangular Plate with Opening by Finite Difference Method. American Journal of Civil and Architecture. Vol.3, No. 5, pp 165-173.

[4] O.M. Ibearugbulem, J.C. Ezeh, U.O. Okpara (2014). Fourth-order Improved Finite Difference Approach to Pure Bending Analysis of Line Continuum. International Journal of Research in Engineering and Technology. eISSN 1219-1163/pISSN 2321-7308.

[5] Szilard, R (2004). Theories and Application of Plate Analysis. New Jersey, John Wiley and Sons Inc.

[6] Timoshenko, S and Woinowsky-Krieger, S (1987). Theory of Plates and Shells. 2nd Edition, McGraw-Hill Book.

[7] Ugural, A.C (1999). Stresses in Plates and Shells (2nd Ed). Singapore, McGraw-Hill.

[8] Ventsel, E. And Krauthammer, K (2001). Thin Plates and Shells, Marcel Decker. Inc. New York.